

On Formation of Singularity of Spherically Symmetric Nonbarotropic Flows

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Abstract

We study an initial boundary value problem on a ball for the heat-conductive system of compressible Navier-Stokes-Fourier equations, in particular, a criterion of breakdown of the classical solution. For smooth initial data away from vacuum, it is proved that the classical solution which is spherically symmetric loses its regularity in a finite time if and only if the **density** *concentrates* or *vanishes* or the **velocity** becomes unbounded around the center. One possible situation is that a vacuum ball appears around the center and the density may concentrate on the boundary of the vacuum ball simultaneously.

1 Introduction and main results

We are concerned with the heat-conductive system of compressible Navier-Stokes-Fourier equations which reads as

$$\begin{cases} \rho_t + \nabla \cdot (\rho U) = 0, \\ (\rho U)_t + \nabla \cdot (\rho U \otimes U) + \nabla P = \mu \Delta U + (\mu + \lambda) \nabla (\nabla \cdot U), \\ c_V ((\rho \theta)_t + \nabla \cdot (\rho \theta U)) + P \nabla \cdot U = \kappa \Delta \theta + \Psi[\nabla U], \end{cases} \quad (1.1)$$

where

$$\Psi[\nabla U] = 2\mu(\mathcal{D}(U))^2 + \lambda(\nabla \cdot U)^2, \quad \mathcal{D}(U) = \frac{\nabla U + \nabla U^t}{2} \quad (1.2)$$

and $t \geq 0, x \in \Omega \subset \mathbb{R}^N (N = 2, 3)$, $\rho = \rho(t, x)$, $U = U(t, x)$ and $\theta = \theta(t, x)$ are the density, fluid velocity and temperature respectively. $P = P(\rho, \theta)$ is the pressure given by a state equation

$$P(\rho) = R\rho\theta. \quad (1.3)$$

The shear viscosity μ , the bulk one λ and heat conductivity κ are constants satisfying the physical hypothesis

$$\mu, \kappa > 0, \quad \mu + \frac{N}{2}\lambda \geq 0. \quad (1.4)$$

The domain Ω is a bounded ball with a radius b , namely,

$$\Omega = B_b = \{x \in \mathbb{R}^N; |x| \leq b < \infty\}. \quad (1.5)$$

We study an initial boundary value problem for (1.1) with the initial condition

$$(\rho, U, \theta)(0, x) = (\rho_0, U_0, \theta_0)(x), \quad x \in \Omega, \quad (1.6)$$

and the boundary condition

$$U = 0, \quad \frac{\partial \theta}{\partial n} = 0 \quad x \in \partial\Omega, \quad \vec{n} \text{ is outnormal vector.} \quad (1.7)$$

We are looking for the smooth spherically symmetric solution (ρ, U) of the problem (1.1), (1.5), (1.6) which enjoys the form

$$\rho(t, x) = \rho(t, |x|), \quad U(t, x) = u(t, |x|) \frac{x}{|x|}, \quad \theta(t, x) = \theta(t, |x|). \quad (1.8)$$

Then, for the initial data to be consistent with the form (1.7), we assume the initial data (ρ_0, U_0) also takes the form

$$\rho_0 = \rho_0(|x|), \quad U_0 = u_0(|x|) \frac{x}{|x|}, \quad \theta_0 = \theta_0(|x|). \quad (1.9)$$

In this paper, we further assume the initial density is uniformly positive, that is,

$$\rho_0 = \rho_0(|x|) \geq \underline{\rho} > 0, \quad x \in \Omega \quad (1.10)$$

for a positive constant $\underline{\rho}$.

Here, it is noted that since the assumption (1.7) implies

$$U(t, x) + U(t, -x) = 0, \quad x \in \Omega. \quad (1.11)$$

We necessarily have $U(t, 0) = 0$ (also $U_0(0) = 0$) as long as classical solutions are concerned.

There are many results about the existence of local and global strong solutions in time of the isentropic system of compressible Navier-Stokes equations when the initial density is uniformly positive (refer to [1, 9–11, 19–21, 24, 25] and their generalization [15–17, 23] to the full system including the conservation law of energy). On the other hand, for the initial density allowing vacuum, the local well-posedness of strong solutions of the isentropic and heat-conductive system was established by Kim [12, 13]. For strong solutions with spatial symmetries, the authors in [14] proved the global existence of radially symmetric strong solutions of the isentropic system in an annular domain, even allowing vacuum initially.

However, it still remains open whether there exist global strong solutions which are spherically symmetric in a ball. The main difficulties lie on the lack of estimates of the density and velocity near the center. In the case vacuum appears, it is worth noting that Xin [26] established a blow-up result which shows that if the initial density has a compact support, then any smooth solution to the Cauchy problem of the full system of compressible Navier-Stokes equations without heat conduction blows up in a finite time, see more generalization in [2], [27]. The same blowup phenomenon occurs also for the isentropic system. Indeed, Zhang-Fang ([28], Theorem 1.8) showed that if $(\rho, U) \in C^1([0, T]; H^k)$ ($k > 3$) is a spherically symmetric solution to the Cauchy problem with the compact supported initial density, then the upper limit of T must be finite. To deal with large discontinuous data, Hoff [3] established global weak solutions of the symmetric compressible heat-conductive flows. On the other hand, it's unclear whether the strong (classical) solutions lose their regularity in a finite time when the initial density is uniformly away from vacuum. Therefore, it is important to study the mechanism of possible blowup of smooth solutions, which is a main issue in this paper.

In the spherical coordinates, the original system (1.1) under the assumption (1.8) takes the form

$$\begin{cases} \rho_t + (\rho u)_\xi = 0, \\ (\rho u)_t + (\rho u^2)_\xi + P_r = \nu u_{\xi r}, \\ c_V((\rho\theta)_t + (\rho u\theta)_\xi) + Pu_\xi = \kappa\theta_{r\xi} + \nu(u_\xi)^2 - \frac{2(N-1)\mu}{r^{N-1}}(r^{N-2}u^2)_r, \end{cases} \quad (1.12)$$

where

$$\nu = 2\mu + \lambda, \quad \frac{\partial}{\partial \xi} = \frac{\partial}{\partial r} + \frac{N-1}{r}. \quad (1.13)$$

Without lossing of generality, we assume $c_V = 1$ and $N = 3$.

Now, we consider the following Lagrangian transformation:

$$t = t, \quad h = \int_0^r \rho(t, s) s^2 ds, \quad \eta = (\rho r^2)^{-1}. \quad (1.14)$$

Then, it follows from (1.12) that

$$h_t = -\frac{u}{\eta}, \quad r_t = u, \quad r_h = \eta, \quad (1.15)$$

and the system (1.12) can be further reduced to

$$\begin{cases} (r^2\eta)_t = (r^2u)_h, \quad (\Longleftrightarrow \eta_t = u_h), \\ u_t = r^2 \left(-R \frac{\theta}{r^2\eta} + \nu \left(\frac{u_h}{r_h} + \frac{2}{r}u \right)_h \right), \\ \left(u_t + Rr^2 \left(\frac{\theta}{r^2\eta} \right)_h = \nu r^2 \left(\frac{(r^2u)_h}{r^2\eta} \right)_h \right) \\ \theta_t = -R\theta \frac{(r^2u)_h}{r^2\eta} + \nu r^2\eta \left(\frac{u_h}{r_h} + \frac{2u}{r} \right)_h^2 - 4\mu(ru^2)_h + \kappa \left(\frac{r^2\theta_h}{r_h} \right)_h \\ = -R\theta \frac{(r^2u)_h}{r^2\eta} + \lambda r^2\eta \left(\frac{u_h}{r_h} + \frac{2}{r}u \right)_h^2 + 2\mu r^2\eta \left(\frac{u_h^2}{r_h^2} + \frac{2u^2}{r^2} \right) + \kappa \left(\frac{r^2\theta_h}{r_h} \right)_h. \end{cases} \quad (1.16)$$

The initial boundary value problem for system (1.16)

$$\begin{aligned} (u, \eta, \theta)(0, h) &= (u_0, \eta_0, \theta_0), \quad (\eta_0 > 0, \theta_0 > 0), \\ u(t, 0) &= u(t, 1) = 0, \quad \theta_h(t, 0) = \theta_h(t, 1) = 0, \end{aligned} \quad (1.17)$$

where $t \geq 0$, $h \in [0, 1]$ and

$$1 = \int_0^b \rho_0(r) r^2 dr = \int_0^b \rho(t, r) r^2 dr, \quad (1.18)$$

according to the conservation of mass. Note that

$$r(t, 0) = 0, \quad r(t, 1) = b. \quad (1.19)$$

We denote E_0 the initial energy

$$E_0 = \int_0^1 \left\{ \frac{u_0^2}{2} + R(r_0^2\eta_0 - \log r_0^2\eta_0 - 1) + (\theta_0 - \log \theta_0 - 1) \right\} dh. \quad (1.20)$$

Our main result is stated as follows.

Theorem 1.1 Assume that the initial data (ρ_0, U_0, θ_0) satisfy (1.8), (1.9), (1.10) and

$$(\rho_0, U_0, \theta_0) \in H^3(\Omega). \quad (1.21)$$

Let (ρ, U, θ) be a classical spherically symmetric solution to the initial boundary value problem (1.1), (1.5), (1.6), (1.7) in $[0, T] \times \Omega$, and T^* be the upper limit of T , that is, the maximal time of existence of the classical solution. Then, if $T^* < \infty$, it holds that

$$\lim_{(t, |x|) \rightarrow (T^*, 0)} \sup \left(\rho(t, |x|) + \frac{1}{\rho}(t, |x|) + |U|(t, |x|) \right) = \infty. \quad (1.22)$$

Remark 1.1 The local existence of smooth solution with initial data as in Theorem 1.1 is classical and can be found, for example, in [13] and references therein. So the maximal time T^* is well defined.

Remark 1.2 There are several results on the blowup criterion for strong and classical solutions to the isentropic and heat-conductive system (1.1) (refer to [4–7, 22, 29] and references therein). Especially, the authors in [7] established the following Serrin-type blowup criterion:

$$\limsup_{T \nearrow T^*} (\|\rho\|_{L^\infty(0, T; L^\infty)} + \|U\|_{L^r(0, T; L^s)}) = \infty, \quad (1.23)$$

for any $r \in [2, \infty]$ and $s \in (3, \infty]$ satisfying

$$\frac{2}{r} + \frac{3}{s} \leq 1. \quad (1.24)$$

Theorem 1.1 asserts that the formation of singularity is only due to the concentration or cavitation of the density and velocity around the center. More precisely, the density anywhere away from the center is bounded up to the maximal time. Also recall that

$$U(t, 0) = 0, \quad \text{for all } t < T^* \quad (1.25)$$

as far as classical solution is concerned. It indicates the possible loss of regularity of velocity at the center.

Remark 1.3 Theorem 1.1 may be viewed as an extension of recent work in [8] where the authors established blowup criterion for barotropic spherically symmetric Navier-Stokes equations.

2 Proof of Theorem 1.1

We only prove the case when $N = 3$ since the case $N = 2$ is even simpler. Throughout of this section, we assume that (ρ, U, θ) is a classical spherically symmetric solution with the form (1.8) to the initial boundary value problem (1.1), (1.5), (1.6), (1.7) in $[0, T] \times \Omega$, and the maximal time T^* , the upper limit of T , is finite, and we denote by C generic positive constants only depending on the initial data and the maximal time T^* .

By simple calculations, we have the following estimates.

Lemma 2.1 *It holds that,*

$$\int_0^1 \eta dh = \int_0^1 \eta_0 dh, \quad (2.1)$$

$$\int_0^1 r^2 \eta dh = \int_0^1 (r^2 \eta)_0 dh = \frac{b^3}{3}, \quad (2.2)$$

$$\int_0^1 \left(\frac{1}{2} u^2 + \theta \right) dh = \int_0^1 \left(\frac{1}{2} u_0^2 + \theta_0 \right) dh. \quad (2.3)$$

Also, we have the following basic energy equality.

Lemma 2.2 *It holds that*

$$\mathcal{E}(t) + \int_0^t \mathcal{V}(\tau) d\tau = \mathcal{E}(0), \quad (2.4)$$

where

$$\begin{aligned} \mathcal{E}(t) &= \int_0^1 \left\{ \frac{u^2}{2} + R(r^2 \eta - \log r^2 \eta - 1) + (\theta - \log \theta - 1) \right\} dh, \\ \mathcal{V}(t) &= \int_0^1 \left\{ \frac{\lambda r^2 \eta}{\theta} \left(\frac{u_h}{r_h} + \frac{2}{r} u \right)^2 + 2 \frac{\mu r^2 \eta}{\theta} \left(\frac{u_h^2}{r_h^2} + \frac{2u^2}{r^2} \right) + \kappa \frac{r^2 \theta_h^2}{r_h \theta^2} \right\} dh \\ &= \int_0^1 \left\{ \left(\lambda + \frac{2}{3} \mu \right) \frac{r^2 \eta}{\theta} \left(\frac{u_h}{r_h} + \frac{2}{r} u \right)^2 + \frac{\mu r^2 \eta}{\theta} \left(\frac{u_h}{r_h} - \frac{u}{r} \right)^2 + \kappa \frac{r^2 \theta_h^2}{r_h \theta^2} \right\} dh, \end{aligned} \quad (2.5)$$

or equivalently,

$$\int_{\Omega} \rho S dx + \int_0^t \int_{\Omega} \left(\frac{1}{\theta} \Psi[\nabla U] + \kappa \frac{|\nabla \theta|^2}{\theta^2} \right) dx ds = \int_{\Omega} \rho_0 S_0 dx, \quad (2.6)$$

where

$$S = R\Phi(\rho^{-1}) + \Phi(\theta) + \frac{1}{2}|U|^2, \quad \Phi(s) = s - \log s - 1. \quad (2.7)$$

In order to prove Theorem 1.1, we can show the following stronger characterization of blowup criterion, that is,

$$\limsup_{(t,h) \rightarrow (T^*, 0)} \left(\rho(t, h) + \frac{1}{\rho}(t, h) + \|U\|_{L^2(t, T^*; L^\infty(B_h))} \right) = \infty. \quad (2.8)$$

We argue by contradiction. For original system (1.1), assume there exists a small $r_1, \varepsilon > 0$ and a constant C , such that

$$\rho(t, r) + \frac{1}{\rho}(t, r) + |U|(t, r) \leq C, \quad \text{for } (t, r) \in (T^* - \varepsilon, T^*) \times [0, r_1]. \quad (2.9)$$

Through the Lagrangian transformation (1.14), one immediately conclude that for system (1.16), there exists a small constant $h_1 > 0$, such that

$$\rho(t, h) + \frac{1}{\rho}(t, h) + |U|(t, h) \leq C, \quad \text{for } (t, h) \in (T^* - \varepsilon, T^*) \times [0, h_1]. \quad (2.10)$$

Denote

$$h_0 = \frac{1}{2} h_1. \quad (2.11)$$

Recall blowup criterion (1.23) by taking $r = \infty$, $s = 2$, it amounts to prove the following Proposition.

Proposition 2.3 For $h_0 = \frac{1}{2}h_1$, there exists a constant C depending on h_0 such that

$$\rho(T, h) + \|U\|_{L^2(0, T; L^\infty(B_h^c))} \leq C(h_0) \quad \text{for } (T, h) \in (T^* - \varepsilon, T^*) \times [h_0, 1]. \quad (2.12)$$

To do that, we prepare the next lemma which gives a relationship between r and y .

Lemma 2.4 There exists a positive constant C independent of T and two strict increasing function $\Theta_i : [0, 1] \rightarrow [0, \infty)$ such that

$$r(t, h) \geq C\Theta_1(h) \quad (2.13)$$

and

$$b^3 - r(t, h)^3 \geq C\Theta_2(h), \quad (2.14)$$

for all $0 \leq t < T^*$.

Proof. For $s \geq 0$, set

$$G(s) = s \log s - s + 1. \quad (2.15)$$

Obviously, G is a convex function in $(0, \infty)$. By Jensen's inequality, one has

$$G\left(\frac{\int_{B_r} \rho dx}{|B_r|}\right) \leq \frac{\int_{B_r} G(\rho) dx}{|B_r|} \iff G\left(\frac{C_0 h}{r^3}\right) \leq \frac{C_1 \int_{\Omega} G(\rho) dx}{r^3}. \quad (2.16)$$

Consequently, the uniform estimates for $r(t, h)$ follows immediately from Entropy inequality (2.6) and (2.16). That is, given $h > 0$ there is a strict increasing function $\Theta_1(h)$ such that

$$r(t, h) \geq \Theta_1(h) \quad \Theta_1(0) = 0 \text{ and } \Theta_1(h) > 0 \text{ for } h > 0. \quad (2.17)$$

By a similar step, one can conclude (2.14) by the following Jensen's inequality

$$G\left(\frac{\int_{B_r^c} \rho dx}{|B_r^c|}\right) \leq \frac{\int_{B_r^c} G(\rho) dx}{|B_r^c|}. \quad (2.18)$$

We are now in a position to establish the pointwise estimates of the density away from the center. To do that, we first write the density in the following form. One may refer to [18] for a similar representation in one-dimensional case.

Lemma 2.5

$$r^2 \eta(t, h) = \frac{1}{\mathcal{B}(t, h) \mathcal{Y}(t, h)} \left((r^2 \eta)_0(h) + \int_0^t \frac{R}{\nu} \mathcal{B}(\tau, h) \mathcal{Y}(\tau, h) \theta(\tau, h) d\tau \right). \quad (2.19)$$

Here

$$\begin{aligned} \mathcal{B}(t, h) = & \exp \frac{1}{\nu} \left\{ \int_{h_0}^h \frac{u_0}{r_0^2} d\xi - \int_{h_0}^h \frac{u}{r^2} d\xi - \left(\int_{h_0}^1 r^2 \eta dh \right)^{-1} \right. \\ & \times \left. \left(\int_{h_0}^1 (r^2 \eta)_0 \int_{h_0}^h \frac{u_0}{r_0^2} d\xi dh - \int_{h_0}^1 (r^2 \eta) \int_{h_0}^h \frac{u}{r^2} d\xi dh + \nu \int_{h_0}^1 (r^2 \eta) dh - \nu \int_{h_0}^1 (r^2 \eta)_0 dh \right) \right\} \end{aligned} \quad (2.20)$$

and

$$\begin{aligned} \mathcal{Y}(t, h) = \exp \left\{ \frac{1}{\nu} \int_0^t \left\{ - \int_{h_0}^h \frac{2u^2}{r^3} d\xi + \left(\int_{h_0}^1 r^2 \eta dh \right)^{-1} \right. \right. \\ \left. \left. \times \left\{ \int_{h_0}^1 \left(u^2 + R\theta + r^2 \eta \int_{h_0}^h \frac{2u^2}{r^3} d\xi \right) dh + \left(\int_0^{h_0} (r^2 u)_h dh \right) \left(\int_0^\tau \sigma(s, h_0) ds \right) \right\} \right\} d\tau \right\}, \end{aligned} \quad (2.21)$$

where

$$\sigma(t, h) = -R \frac{\theta}{r^2 \eta} + \nu \frac{(r^2 \eta)_t}{r^2 \eta}. \quad (2.22)$$

Proof. In view of (1.16), it holds

$$\frac{1}{r^2} u_t = \left(-R \frac{\theta}{r^2 \eta} + \nu \frac{(r^2 \eta)_t}{r^2 \eta} \right)_h \triangleq (\sigma(t, h))_h. \quad (2.23)$$

Thus, for $h > h_0 > 0$, integrating (2.23) over (h_0, h) , we deduce that

$$\left(\int_{h_0}^h \frac{u}{r^2} d\xi \right)_t + \int_{h_0}^h \frac{2u^2}{r^3} d\xi = \sigma(t, h) - \sigma(t, h_0). \quad (2.24)$$

Multiplying $\frac{r^2 \eta}{\nu}$ on both sides of (2.24) yields

$$\frac{r^2 \eta}{\nu} \left\{ \left(\int_{h_0}^h \frac{u}{r^2} d\xi \right)_t + \int_{h_0}^h \frac{2u^2}{r^3} d\xi + \sigma(t, h_0) \right\} = -\frac{R}{\nu} \theta + (r^2 \eta)_t, \quad (2.25)$$

which is

$$(r^2 \eta)_t - \frac{1}{\nu} \left\{ \left(\int_{h_0}^h \frac{u}{r^2} d\xi \right)_t + \int_{h_0}^h \frac{2u^2}{r^3} d\xi + \sigma(t, h_0) \right\} (r^2 \eta) = \frac{R}{\nu} \theta. \quad (2.26)$$

Denoted by

$$\mathcal{A}(t, h) = -\frac{1}{\nu} \left\{ \left(\int_{h_0}^h \frac{u}{r^2} d\xi \right)_t + \int_{h_0}^h \frac{2u^2}{r^3} d\xi + \sigma(t, h_0) \right\}. \quad (2.27)$$

In view of (2.26) and (2.27), one has

$$(r^2 \eta)(t, h) = \exp \left(- \int_0^t \mathcal{A} d\tau \right) \cdot \left\{ (r^2 \eta)_0(h) + \int_0^t \exp \left(\int_0^\tau \mathcal{A} ds \right) \cdot \frac{R}{\nu} \theta d\tau \right\}. \quad (2.28)$$

On the other hand, recall (2.23)-(2.24)

$$\sigma(t, h_0) = - \left(\int_{h_0}^h \frac{u}{r^2} d\xi \right)_t - \int_{h_0}^h \frac{2u^2}{r^3} d\xi - R \frac{\theta}{r^2 \eta} + \nu \frac{(r^2 \eta)_t}{r^2 \eta}. \quad (2.29)$$

Multiplying $r^2 \eta$ on both sides of (2.29), the first term becomes

$$\begin{aligned} -(r^2 \eta) \left(\int_{h_0}^h \frac{u}{r^2} d\xi \right)_t &= - \left\{ (r^2 \eta) \int_{h_0}^h \frac{u}{r^2} d\xi \right\}_t + (r^2 \eta)_t \int_{h_0}^h \frac{u}{r^2} d\xi \\ &= - \left\{ (r^2 \eta) \int_{h_0}^h \frac{u}{r^2} d\xi \right\}_t + \left\{ (r^2 u) \int_{h_0}^h \frac{u}{r^2} d\xi \right\}_h - (r^2 u) \frac{u}{r^2}. \end{aligned} \quad (2.30)$$

Integrating (2.29) on $[h_0, 1] \times [0, t]$ and taking into account the boundary conditions (1.17) yields

$$\begin{aligned} \int_0^t \int_{h_0}^1 r^2 \eta \sigma(\tau, h_0) dh d\tau &= - \int_0^t \int_{h_0}^1 \left\{ (r^2 \eta) \int_{h_0}^h \frac{u}{r^2} d\xi \right\}_t dh d\tau + \int_0^t \int_{h_0}^1 \left\{ (r^2 u) \int_{h_0}^h \frac{u}{r^2} d\xi \right\}_h dh d\tau \\ &\quad - \int_0^t \int_{h_0}^1 \left\{ u^2 + R\theta + (r^2 \eta) \int_{h_0}^h \frac{2u^2}{r^3} d\xi \right\} dh d\tau \\ &\quad + \nu \int_0^t \int_{h_0}^1 (r^2 u)_h dh d\tau. \end{aligned} \quad (2.31)$$

Therefore,

$$\begin{aligned} \int_0^t \int_{h_0}^1 r^2 \eta \sigma(\tau, h_0) dh d\tau &= \int_{h_0}^1 \left((r^2 \eta)_0 \cdot \int_{h_0}^h \frac{u_0}{r_0^2} d\xi \right) dh - \int_{h_0}^1 \left((r^2 \eta) \cdot \int_{h_0}^h \frac{u}{r^2} d\xi \right) dh \\ &\quad - \int_0^t \int_{h_0}^1 \left\{ u^2 + R\theta + (r^2 \eta) \int_{h_0}^h \frac{2u^2}{r^3} d\xi \right\} dh d\tau \\ &\quad + \nu \int_{h_0}^1 (r^2 \eta) dh - \nu \int_{h_0}^1 (r^2 \eta)_0 dh. \end{aligned} \quad (2.32)$$

In view of boundary condition and (1.16)₁, the l.h.s of (2.32) can be written as

$$\begin{aligned} &\int_0^t \left(\int_{h_0}^1 r^2 \eta dh \right) \left(\int_0^\tau \sigma(s, h_0) ds \right)' d\tau \\ &= \left(\int_{h_0}^1 r^2 \eta dh \right) \left(\int_0^t \sigma(\tau, h_0) d\tau \right) - \int_0^t \left(\int_{h_0}^1 (r^2 \eta)_t dh \right) \left(\int_0^\tau \sigma(s, h_0) ds \right) d\tau \\ &= \left(\int_{h_0}^1 r^2 \eta dh \right) \left(\int_0^t \sigma(\tau, h_0) d\tau \right) + \int_0^t \left(\int_{h_0}^{h_0} (r^2 u)_h dh \right) \left(\int_0^\tau \sigma(s, h_0) ds \right) d\tau. \end{aligned} \quad (2.33)$$

Collecting (2.28)-(2.33), the proof of Lemma 2.5 is completed.

We immediately have the following corollary

Corollary 2.6 *Given $0 < h_0 < 1$, for $h_0 \leq h \leq 1$, there exists a constant C depending on h_0 such that*

$$C^{-1}(h_0) \leq \mathcal{B}(t, h), \mathcal{Y}(t, h) \leq C(h_0), \quad (2.34)$$

and

$$C^{-1}(h_0) \exp\{C^{-1}(h_0)(t - \tau)\} \leq \frac{\mathcal{Y}(t, h)}{\mathcal{Y}(\tau, h)} \leq C(h_0) \exp\{C(h_0)(t - \tau)\}, \quad 0 \leq \tau < t. \quad (2.35)$$

Proof. First,

$$\int_{h_0}^1 r^2 \eta dh = \int_{h_0}^1 \frac{1}{\rho} dh = \int_{r(h_0)}^b r^2 dr = \frac{b^3 - r(h_0)^3}{3}. \quad (2.36)$$

In view of (2.14), one gets

$$0 < C\Theta_2(h_0) \leq \int_{h_0}^1 r^2 \eta dh \leq b^3. \quad (2.37)$$

Similarly,

$$0 \leq \int_{h_0}^1 \frac{u^2}{r^3} d\xi = \int_{r(h_0)}^b \frac{\rho u^2 r^2}{r^3} dr \leq C r(h_0)^{-3} \leq C \Theta_1(h_0)^{-3}. \quad (2.38)$$

$$\begin{aligned} \left| \int_{h_0}^1 \frac{u}{r^2} dh \right| &\leq \min_{h \geq h_0} r(t, h)^{-2} \int_{h_0}^1 |u| dh \leq C \Theta_1(h_0)^{-2} \left(\int_0^1 u^2 dh \right)^{1/2} (1 - h_0)^{1/2} \\ &\leq C \Theta_1(h_0)^{-2}. \end{aligned} \quad (2.39)$$

Also, one can verify

$$\int_{h_0}^1 r^2 \eta \int_{h_0}^h \frac{2u^2}{r^3} d\xi dh \quad (2.40)$$

is bounded.

Observing that

$$\begin{aligned} \int_{h_0}^1 r^2 \eta \int_{h_0}^h \frac{2u^2}{r^3} d\xi dh &= \int_{h_0}^1 \left\{ \left(\frac{r^3}{3} \int_{h_0}^h \frac{2u^2}{r^3} d\xi \right)_h - \frac{r^3}{3} \frac{2u^2}{r^3} \right\} dh \\ &= \frac{b^3}{3} \int_{h_0}^1 \frac{2u^2}{r^3} d\xi - \int_{h_0}^1 \frac{2u^2}{3} dh \end{aligned} \quad (2.41)$$

is bounded from below and above by a constant $C(h_0)$.

To finish the proof, it suffices to bound

$$\int_0^t \sigma(\tau, h_0) d\tau. \quad (2.42)$$

Recall that the r.h.s of (2.32) is bounded by some constant $C(h_0)$ and (2.33), one has

$$\left| \int_0^t \sigma(\tau, h_0) d\tau \right| \leq C(h_0) + C(h_0) \int_0^t \left(\left| \int_0^{h_0} (r^2 u)_h dh \right| \right) \left(\left| \int_0^\tau \sigma(s, h_0) ds \right| \right) d\tau. \quad (2.43)$$

On the other hand, with the help of (2.10) yields that

$$\begin{aligned} \int_0^{h_0} (r^2 u)_h dh &= \int_0^{h_0} \left(r^3 \frac{u}{r} \right)_h dh \\ &= \int_0^{h_0} 3r\eta u dh + \int_0^{h_0} r^3 \left(\frac{u}{r} \right)_h dh \\ &\leq C \max_{0 \leq h \leq h_0} |r^2 \eta|^{\frac{1}{2}} \left(\int_0^{h_0} \eta dh \right)^{\frac{1}{2}} \left(\int_0^{h_0} u^2 dh \right)^{\frac{1}{2}} \\ &\quad + \left(\int_0^{h_0} \frac{r^4}{\eta \theta} \left(\frac{u}{r} \right)_h^2 \right)^{\frac{1}{2}} \left(\int_0^{h_0} r^2 \eta \theta dh \right)^{\frac{1}{2}} \\ &\leq C(h_0) + C(h_0) \left(\int_0^{h_0} \frac{r^4}{\eta \theta} \left(\frac{u}{r} \right)_h^2 \right)^{\frac{1}{2}} \\ &= C(h_0) + C(h_0) \left(\int_0^{h_0} \frac{r^2 \eta}{\theta} \left(\frac{u_h}{r_h} - \frac{u}{r} \right)^2 \right)^{\frac{1}{2}} \\ &\leq C(h_0) + C(h_0) \mathcal{V}(t)^{\frac{1}{2}}, \end{aligned} \quad (2.44)$$

where we used energy inequality, (2.1), (2.3) and (2.46) in the next Lemma 2.7.

Consequently, the desired bound for $\sigma(\tau, h_0)$ follows immediately from Gronwall's inequality and (2.43)-(2.44).

Thus finishes the proof of Corollary 2.6.

Hence, substituting (2.20) and (2.21) into (2.19), we finally arrive at

Lemma 2.7

$$C^{-1}(h_0) \leq \rho(t, h) \leq C(h_0), \quad 0 < h_0 \leq h, \quad (2.45)$$

$$C^{-1} \leq \int_0^1 \theta dh \leq C, \quad (2.46)$$

and

$$\int_0^t \max_{h \in [h_0, 1]} u^2(\tau, h) d\tau \leq C(h_0). \quad (2.47)$$

Remark 2.1 Theorem 1.1 follows immediately from Lemma 2.7.

Proof. The right-hand side of (2.45) is a direct consequence of the fact that $\theta > 0$, (2.19) and (2.34). It remains to show the upper bound of $r^2\eta = \frac{1}{\rho}$.

Step 1. Multiplying $\frac{1}{\theta}$ on both sides of (1.16)₃ yield

$$\begin{aligned} (\log \theta)_t &= -R (\log(r^2\eta))_t + \left(\lambda + \frac{2}{3}\mu\right) \frac{r^2\eta}{\theta} \left(\frac{u_h}{r_h} + \frac{2}{r}u\right)^2 + \frac{\mu r^2\eta}{\theta} \left(\frac{u_h}{r_h} - \frac{u}{r}\right)^2 \\ &\quad + \kappa \left(\frac{1}{\theta} \cdot \frac{r^2\theta_h}{r_h}\right)_h + \kappa \frac{\theta_h}{\theta^2} \cdot \frac{r^2\theta_h}{r_h}. \end{aligned} \quad (2.48)$$

Integrating (2.48) over $[0, 1] \times [0, t]$ and recall (1.4), (1.17) to obtain

$$\left\{ \int_0^1 \log \theta dh - \int_0^1 \log \theta_0 dh \right\} \leq -R \left\{ \int_0^1 \log(r^2\eta) dh - \int_0^1 \log(r^2\eta)_0 dh \right\}. \quad (2.49)$$

Applying Jensen's inequality to (2.49) and (2.2) to arrive

$$\begin{aligned} \log \int_0^1 \theta dh &\geq \int_0^1 \log \theta dh \\ &\geq \int_0^1 \log \theta_0 dh - R \int_0^1 \log(r^2\eta) dh + R \int_0^1 \log(r^2\eta)_0 dh \\ &\geq \int_0^1 \{ \log \theta_0 + R \log(r^2\eta)_0 \} dh - R \log \int_0^1 r^2\eta dh \\ &= C_1, \end{aligned} \quad (2.50)$$

which gives the desired bound for $\int_0^1 \theta dh$.

Step 2. Applying the mean value theorem to continuous function $\theta(t, h)$ to get

$$\forall t > 0, \exists h(t) \in [h_0, 1], \quad s.t. \quad \theta(t, h(t)) = \frac{\int_{h_0}^1 \theta(t, h) dh}{1 - h_0}. \quad (2.51)$$

Therefore, for $h \geq h_0$

$$\begin{aligned}
\theta(t, h)^{\frac{1}{2}} &= \theta(t, h(t))^{\frac{1}{2}} + \int_{h(t)}^h \frac{\theta_h}{2\theta(t, \xi)^{\frac{1}{2}}} d\xi \\
&\leq C(1 - h_0)^{-1} \left(\int_0^1 \theta dh \right)^{\frac{1}{2}} + \left(\int_{h_0}^1 \theta d\xi \right)^{\frac{1}{2}} \left(\int_{h_0}^1 \frac{\theta_h^2}{4\theta^2} d\xi \right)^{\frac{1}{2}} \\
&\leq C \left\{ 1 + \max_{h \in [h_0, 1]} r^2 \eta(t, h) \int_{h_0}^1 \frac{\theta_h^2}{r^2 \eta \theta^2} d\xi \right\}^{\frac{1}{2}} \\
&\leq C(h_0) \left\{ 1 + \max_{h \in [h_0, 1]} r^2 \eta(t, h) \cdot \mathcal{V}(t) \right\}^{\frac{1}{2}}.
\end{aligned} \tag{2.52}$$

Consequently,

$$\theta(t, h) \leq C(h_0) \left\{ 1 + \max_{h \in [h_0, 1]} r^2 \eta(t, h) \cdot \mathcal{V}(t) \right\}, \quad h_0 \leq h \leq 1. \tag{2.53}$$

Step 3. Observing that

$$\begin{aligned}
r^2 \eta(t, h) &= \frac{1}{\mathcal{B}(t, h) \mathcal{Y}(t, h)} \left((r^2 \eta)_0(h) + \int_0^t \frac{R}{\nu} \mathcal{B}(\tau, h) \mathcal{Y}(\tau, h) \theta(\tau, h) d\tau \right) \\
&\leq \frac{1}{\mathcal{B}(t, h) \mathcal{Y}(t, h)} (r^2 \eta)_0(h) \\
&\quad + C \int_0^t \frac{\mathcal{B}(\tau, h)}{\mathcal{B}(t, h)} \cdot \frac{\mathcal{Y}(\tau, h)}{\mathcal{Y}(t, h)} \left\{ 1 + \max_{h \in [h_0, 1]} r^2 \eta(t, h) \cdot \mathcal{V}(t) \right\} d\tau.
\end{aligned} \tag{2.54}$$

Hence,

$$\begin{aligned}
\max_{h \in [h_0, 1]} r^2 \eta(t, h) &\leq C + C \int_0^t \exp\{-\alpha(t - \tau)\} \left\{ 1 + \max_{h \in [h_0, 1]} r^2 \eta(\tau, h) \cdot \mathcal{V}(\tau) \right\} d\tau \\
&\leq C + C \int_0^t \exp\{-\alpha(t - \tau)\} \max_{h \in [h_0, 1]} r^2 \eta(\tau, h) \cdot \mathcal{V}(\tau) d\tau.
\end{aligned} \tag{2.55}$$

Write

$$E(t) = \int_0^t \exp\{-\alpha(t - \tau)\} \max_{h \in [h_0, 1]} r^2 \eta(\tau, h) \cdot \mathcal{V}(\tau) d\tau. \tag{2.56}$$

One immediately has

$$\begin{aligned}
E_t &\leq \max_{h \in [h_0, 1]} r^2 \eta(t, h) \cdot \mathcal{V}(t) \leq (C + CE) \mathcal{V}(t) - \alpha E, \\
E_t + (\alpha - C \mathcal{V}(t)) E &\leq C \mathcal{V}(t),
\end{aligned} \tag{2.57}$$

Applying Gronwall's inequality to (2.57) yields

$$E \leq C \exp \left\{ - \int_0^t \alpha - C \mathcal{V}(\tau) d\tau \right\} \times C \int_0^t \exp \left\{ \int_0^\tau \alpha - C \mathcal{V}(s) ds \right\} \mathcal{V}(\tau) d\tau. \tag{2.58}$$

The upper bound of $r^2 \eta$ follows from (2.55) and (2.58).

Step 4. It suffices to establish a bound for the velocity. Indeed,

$$\begin{aligned}
\left(\frac{u}{r} \right)^2(h) &\leq \left\{ \int_{h_0}^h \left| \left(\frac{u}{r} \right)_h \right| \right\}^2 \leq \int_{h_0}^h \frac{r^4}{\theta r_h} \left(\frac{u}{r} \right)_h^2 dh \cdot \int_{h_0}^h \frac{\theta r_h}{r^4} dh \\
&\leq C(h_0) \int_{h_0}^1 \frac{r^2 \eta}{\theta} \left(\frac{u_h}{r_h} - \frac{u}{r} \right)^2 dh.
\end{aligned} \tag{2.59}$$

That is

$$\max_{h \in [h_0, 1]} \left(\frac{u}{r} \right)^2(h) \leq C(h_0) \mathcal{V}(t) \in L^1(0, T). \quad (2.60)$$

To conclude, (2.47) is a direct consequence of (2.60).

This finishes the proof of Lemma 2.7.

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